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Linear stability of a non-symmetric, inviscid, Kármán street of small uniform vortices

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The classical point-vortex model for a Kármán vortex street is linearly stable only for a single, isolated, marginally stable, case. This property has been shown numerically to hold for streets formed by symmetric rows of uniform vortices of equal area. That result is extended here to the case in which the areas of the vortices in the two rows are not necessarily equal. The method used is an analytic perturbation valid when the vortex areas are small, and applied using an automatic symbolic manipulator.

1. Introduction

The regular arrangement of vortices sometimes found in the wake of bluff bodies was first modelled by von Kármán (Lamb 1945) as a double infinite street of point vortices, moving with respect to the fluid at infinity at some constant velocity, and whose properties are characterized by a single parameter, H, which is the ratio between the separation of the two rows and the wavelength of the periodic vortex array. The stability of this arrangement to perturbations in the positions of the vortices was studied by von Kármán by expanding those perturbations in terms of normal modes, parameterized by a wavenumber κ. The stability of a given mode depends both on κ and H, and a given street is stable when all the modes, κ, are stable for the given fixed value of H. It turns out that vortex streets are generally unstable, containing an unstable range of wavenumbers for all values of H except for one, $H_c = 0.28055\ldots$, for which all wavenumbers are neutrally stable. It was shown later that, even in this case, the street exhibits a nonlinear instability (Domm 1956), due in essence to the fact that any small perturbation 'modifies' H enough to move the street into the unstable regime.

Since natural wakes show regular vortex streets that persist for long times, it has generally been felt that a more complete model should produce a finite stability range of H that would guarantee linear stability for streets contained inside that range. A natural attempt is to consider models formed by arrays of regions of uniform vorticity and finite area, instead of by point vortices (Kida 1982; Saffman & Schatzman 1982). After several unsuccessful attempts, it was shown by Meiron, Saffman & Schatzman (1984, hereinafter referred to as MSS) that the stability behaviour of those models is again the same; at least for moderate values of the area of the individual vortices, there is a single value of H for which the street is stable. In fact, Jiménez (1987) showed that this behaviour is a consequence of the Hamiltonian structure of the system and that it should hold for any inviscid model of the street that retains the back to fore (as opposed to the top to bottom) symmetry of the basic flow, and that, therefore, the persistence of a natural vortex street is most probably due to viscous effects, or to the differences between the spatial development of natural wakes and the temporal evolution model implicity in a periodic vortex street.

Still, the finite area case is interesting in that it provides a natural model for real flows, and a starting point for dissipative models of the wake. Most of the evidence for its behaviour is presently numerical (Saffman & Schatzman 1982; MSS). Kida (1982) attempted an algebraic perturbation expansion for small areas, but his expansion contained algebraic mistakes, and some of his conclusions were in error. Perturbation expansions have the advantage that they provide analytic information on the local behaviour of the system, but they tend to be difficult to implement because of the computational complexity involved. Automatic symbolic manipulation programs have changed that, and it is now possible to develop fairly complex perturbation schemes with an error probability comparable to that of numerical analysis. We present here a perturbation treatment for the finite area model, in the case in which the areas are small. The results generally confirm those of the numerical work, and of the Hamiltonian analysis, and they are extended to the case in which the vortices of the two rows have different areas, but are uniform within each row.

2. Isolated patches of vorticity

We introduce in this section the perturbation method used to describe the motion of each of the uniform patches of vorticity which are assumed to form the vortex street. The method used is similar to the one used in MSS to develop their numerical scheme, and it will only be outlined here, but we will give algebraic expressions and define the notation for the quantities used later in the paper. While the manipulation necessary to derive the equations of motion to the accuracy used in this paper could probably have been done by hand, the probability of error was fairly high, and we preferred to use an automatic symbolic manipulator (Hearn 1983) to do the algebra.

Consider an isolated patch of uniform vorticity, of which the total circulation is γ, and area

$$S = \pi\epsilon^2. \tag{1}$$

We will assume that the 'radius', ϵ, is small with respect to the characteristic lengthscale of the velocity field that would exist in the absence of the patch under study.

Define the complex variable $z = x + \mathrm{i}y$, and consider the complex velocity $\bar{w} = u - \mathrm{i}v$, where the bar stands for complex conjugation. Outside the vortex, the flow is irrotational, and the velocity, $\bar{w}_0$, is an analytic function of z. Inside it, it can be expressed as an irrotational part, plus a particular solution due to the vorticity, as

$$\bar{w}_\mathrm{i}(z) - \bar{w}_\mathrm{c} = -\frac{\mathrm{i}\gamma(\bar{z}-\bar{z}_\mathrm{c})}{2\pi\epsilon^2} + f(z), \tag{2}$$

where z_c is the position of the vortex centre, $w_\mathrm{c} = \mathrm{d}z_\mathrm{c}/\mathrm{d}t$ is its translation velocity, and the function f is analytic in z. At the perimeter of the vortex, the velocity is continuous, and

$$f(z) - \bar{w}_0(z) + \bar{w}_\mathrm{c} = \frac{\mathrm{i}\gamma(\bar{z}-\bar{z}_\mathrm{c})}{2\pi\epsilon^2}. \tag{3}$$

The velocity outside the vortex, measured with respect to the vortex centre, can be expressed as a series in $z-z_c$, as

$$\bar{w}_0(z)-\bar{w}_c = -\bar{w}_s+\frac{\gamma}{2\pi}\sum_{k=2} c_k(z-z_c)^{k-1} - \frac{i\gamma}{2\pi}\sum_{k=1}\frac{d_k\,\epsilon^{2k-2}}{(z-z_c)^k}. \tag{4}$$

The Taylor part of this series reflects the behaviour of the velocity at infinity, where the influence of the vortex patch has died out, and it is just the series expansion, in the vicinity of the vortex, of the externally imposed velocity, $\bar{w}_\infty(z)-\bar{w}_\infty(z_c)$. The 'straining coefficients', c_k, can be simply expressed as

$$c_k = \frac{2\pi}{\gamma(k-1)!}\frac{\partial^{k-1}\bar{w}_\infty}{\partial z^{k-1}}, \tag{5}$$

where the derivatives are evaluated at the vortex centre. The constant term $\bar{w}_s$ is the 'drift' velocity of the vortex with respect to the externally imposed velocity, $\bar{w}_\infty(z_c)$. When the limit of (4) is taken as $z\rightarrow\infty$, the Laurent part of the series disappears, the Taylor part can be expressed in terms of the $\bar{w}_\infty(z)$, and, in the left-hand side, $w_0\rightarrow\bar{w}_\infty(z)$. The result is an expression for the velocity of the vortex centre,

$$\frac{d\bar{z}_c}{dt} = \frac{\gamma}{2\pi}c_1+\bar{w}_s = \bar{w}_\infty(z_c)+\bar{w}_s. \tag{6}$$

The analytic part of the velocity (2) inside the vortex can also be expressed as a Taylor series,

$$f(z) = \frac{i\gamma}{2\pi}\sum_{k=0} f_k(z-z_c)^k, \tag{7}$$

which is valid inside a disk which extends to the innermost singularity of f, which is outside the vortex. As a consequence, this expansion is valid in some part of, but not necessarily in all, the contour of the vortex. The same is true of (4) which is valid outside a similar disk, extending in this case to the outermost singularity of $\bar{w}_0$, which is inside the vortex. If these two series had a common domain of validity, they could be substituted in (3), and used to determine the coefficients of the velocity expansions in terms of the vortex shape. However, because the vortex is not circular, and there is no guarantee that the outermost singularity of $\bar{w}_0$ is farther from the vortex centre than the innermost singularity of $f(z)$, it is not true in general that such a domain exists.

Consider now a conformal mapping which transforms the outside of the vortex patch into the outside of the unit circle,

$$z-z_c = g(\eta,t) = \epsilon\eta\left[b_0+\frac{b_1\epsilon}{\eta}+\frac{b_2\epsilon^2}{\eta^2}+\dots\right]. \tag{8}$$

When this expansion is substituted in the two series for the velocities, the resulting expansion in η for (4) will converge outside a disk which extends to, and presumably goes slightly inside, the unit circumference, which now represents the vortex contour. Similarly, the expansion for (7) holds in an annulus defined on the inside by the domain of validity of (8), and, on the outside, by the closest singularity of $f(z)$, which will presumably be slightly outside the unit circle. As a consequence, there will exist a common annulus in which both expansions are valid, and which contains the unit circumference. Moreover, on that circumference, $\bar{\eta} = 1/\eta$, and

$$\bar{z}-\bar{z}_c = \frac{\epsilon}{\eta}[b_0+\bar{b}_1\epsilon\eta+\bar{b}_2\epsilon^2\eta^2+\dots]. \tag{9}$$

These expressions can be substituted now in (3), giving a series in powers of η which must be satisfied term by term, and which provides enough equations to determine the coefficients d_k and f_k in terms of b_k and $\bar{w}_s$.

Note that the existence of a common annulus of definition for both expansions in η is necessary to ensure that the procedure is valid. This existence is controlled by the shape of the vortex contour, and, in particular, by the outermost singularity of the map $g(\eta, t)$. That map is intended to work outside the vortex and does not necessarily work on its interior. Actually, in general, it does not even map the interior of the vortex into the interior of the unit circle. The distance from the outermost singularity to the unit circumference is controlled mainly by the local radius of curvature of the vortex contour. For example, it can be shown that, when the contour develops a corner, the singularity lies on the circumference itself. Under those circumstances, the annulus of common validity vanishes, and the success of the method is not guaranteed.

Except for those cases, the procedure is, in principle, not restricted to small perturbations of a circular vortex and should give a convergent series representation of the velocity induced by a vortex in terms of its shape. In practice, the algebraic complexity of the equations for the coefficients of the truncated series of a given length grows very rapidly and it is only practical to consider the expansions as asymptotic in ϵ. The powers of ϵ in front of the coefficients of the expansions (7 and 8) reflect this approach and correspond to the maximum order of magnitude for each term, at equilibrium, when the straining coefficients, c_k, are $O(1)$.

There is some freedom in the choice of the map in (8). In particular, it is always possible to make b_0 real, and to adjust one additional coefficient by fixing the arbitrary position of the vortex centre (z_c). We will choose to make the dipole component of the far-field velocity equal to zero.

$$d_2 = 0, \tag{10}$$

which is equivalent to defining the origin of coordinates at the centroid (centre of gravity) of the vortex patch, and is the same definition used in MSS. One extra condition is given by the area of the patch, which can be expressed as

$$S = \pi\epsilon^2 = \pi\epsilon^2[b_0^2 - \sum_{k=2} (k-1)\,\epsilon^{2k}\, b_k \bar{b}_k], \tag{11}$$

and allows us to express b_0 in terms of the rest of the coefficients in the expansion.

The evolution equations for b are found by imposing that the contour of the vortex, or the unit circle in η, is mapped on itself by the flow, which is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\eta\bar{\eta}) = \mathrm{Re}\left(\eta\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}t}\right),$$

or

$$\mathrm{Re}\left[\left(\overline{\eta\frac{\mathrm{d}g}{\mathrm{d}\eta}}\right)\left(w_0 - w_c - \frac{\mathrm{d}g}{\mathrm{d}t}\right)\right] = 0. \tag{12}$$

The result should be seen as a series in η, and solved term by term. It provides values for $\mathrm{d}b_k/\mathrm{d}t$ and for $\bar{w}_s$. Here we give the leading terms of the expansions which will be needed for the subsequent analysis,

$$\bar{w}_0 = \bar{w}_\infty(z) - \frac{\mathrm{i}\gamma}{2\pi}\left[\frac{1}{z} + \frac{\epsilon^4 b_2}{z^3}\right] + O(\epsilon^6), \tag{13}$$

$$\bar{w}_s = \frac{\gamma\epsilon^4}{2\pi} b_2 c_3 + O(\epsilon^6). \tag{14}$$

Also, to this order, $b_1 = 0$. In principle, the drift velocity includes contributions from the third-order harmonic, b_3, which are of the same order of magnitude as those in (14) but, with the choice of coordinates used here, those contributions cancel identically, and it is possible to write the motion of the vortex centroids in terms just of z and b_2. The evolution equations for the coefficients are

$$\frac{2\pi\epsilon^2}{\gamma}\frac{db_2}{dt} = ib_2 + \bar{c}_2 + O(\epsilon^4), \quad \frac{2\pi\epsilon^2}{\gamma}\frac{db_3}{dt} = 2ib_3 + \bar{c}_3 + O(\epsilon^4). \tag{15}$$

3. The non-symmetric vortex street

Consider a vortex street formed by two rows of vortices, such that the centre of the mth vortex in the lower row is at the complex point $z_{m\beta} = m$, while that of the $n$the vortex of the top row is at $z_{n\alpha} = n + \frac{1}{2} + iH$. From now on we will associate the subscripts β and α with the lower and upper rows of vortices respectively. The circulations associated to the vortices in each row are equal but opposite in sign, and we will take them to be $\gamma_\alpha = 2\pi$, $\gamma_\beta = -2\pi$. All the vortices in a given row are equivalent and have non-zero areas, $S_\alpha = \pi\epsilon_\alpha^2$ and $S_\beta = \pi\epsilon_\beta^2$, which we will assume to be different but of the same order of magnitude. We will use later the notation

$$\epsilon_\alpha^2 = (1+Q)\,\epsilon^2, \quad \epsilon_\beta^2 = (1-Q)\,\epsilon^2, \tag{16}$$

where Q is $O(1)$, and $Q = 0$ corresponds to the symmetric street.

At equilibrium, both rows move at a constant common velocity. The velocity induced at the centre of a vortex in row α is the sum of the contributions from all the other vortices in both rows

$$\bar{w}_\alpha = -i\frac{\gamma_\alpha}{2\pi}\sum_{n\neq 0}\left(\frac{1}{n} + \frac{\epsilon_\alpha^4 b_{2\alpha}}{n^3}\right) - i\frac{\gamma_\beta}{2\pi}\sum_m\left(\frac{1}{\frac{1}{2}+iH-m} + \frac{\epsilon_\beta^4 b_{2\beta}}{(\frac{1}{2}+iH-m)^3}\right) + O(\epsilon^6). \tag{17}$$

which can be expressed as

$$\bar{w}_\alpha = -\tau_1(H) + \epsilon_\beta^4 b_{2\beta}\tau_3(H). \tag{18}$$

A similar expression applies to the velocity induced at the centres of the vortices in the lower row.

$$\bar{w}_\beta = -\tau_1(H) + \epsilon_\alpha^4 b_{2\alpha}\tau_3(H). \tag{19}$$

We have used in these expressions the functions

$$\tau_k(H) = i^k \sum_{n=-\infty}^{\infty} (n - \tfrac{1}{2} - iH)^{-k}, \tag{20}$$

which appear in the summation of the contributions from a row. We will need

$$\tau_1 = -\pi\tanh(\pi H), \quad \tau_2 = -\frac{\pi^2}{\cosh^2(\pi H)}, \quad \tau_3 = \frac{\pi^3\tanh(\pi H)}{\cosh^2(\pi H)}. \tag{21}$$

The straining coefficients are computed similarly as

$$c_{2\alpha} = c_{2\beta} = i\,[\tfrac{1}{3}\pi^2 + \tau_2(H)], \quad c_{3\alpha} = -c_{3\beta} = \tau_3(H), \tag{22}$$

and, as a consequence of the equations of motion for b (15), the equilibrium shape of the vortex cores is,

$$b_{2\alpha} = b_{2\beta} \equiv b_2 = \tfrac{1}{3}\pi^2 + \tau_2(H), \quad b_{3\alpha} = -b_{3\beta} = \tfrac{1}{2}\mathrm{i}\tau_3(H). \tag{23}$$

To find the translation velocity of the vortex centres, we have to add to the velocities (18), (19) the self-induced correction (14) which, at equilibrium, is

$$\bar{w}_{s\alpha} = \epsilon_\alpha^4 b_2 \tau_3(H), \quad \bar{w}_{s\beta} = \epsilon_\beta^4 b_2 \tau_3(H). \tag{34}$$

Finally, the common translation velocity is

$$\frac{\mathrm{d}\bar{z}_\alpha}{\mathrm{d}t} = \frac{\mathrm{d}\bar{z}_\beta}{\mathrm{d}t} = -\tau_1(H) + (\epsilon_\alpha^4 + \epsilon_\beta^4)\,\tau_3(H)\,[\tfrac{1}{3}\pi^2 + \tau_2(H)]. \tag{25}$$

Note that there is no *a priori* guarantee that the velocities for both rows should come out to be equal. In fact, this is only true because the self-induced velocity correction for one row exactly compensates the extra induction due to the non-zero area of the other one. Should this compensation not have been exact, the equilibrium condition for the vortex street would have had to include different magnitudes for the vortex circulations, $\gamma_\alpha \neq -\gamma_\beta$, to compensate for the difference in vortex areas. That any equilibrium configuration of the vortex street should satisfy $\gamma_\alpha = -\gamma_\beta$ was proved in MSS on purely kinematic grounds. The derivation in this section is an independent confirmation of that fact, at least to the accuracy of the expansion.

4. Stability equations

To study the stability of the equilibrium configuration, we let the centres of the vortices be perturbed from their equilibrium position by small amounts, which implies corresponding perturbations in the vortex shapes. It is convenient to express the perturbations in terms of periodic modes, with wavenumber κ, defined as

$$\left.\begin{aligned} z_{m\alpha} &= m + \tfrac{1}{2} + \mathrm{i}H + \delta z_\alpha^+ \,\mathrm{e}^{\mathrm{i}\kappa(m+\frac{1}{4})} + \delta z_\alpha^- \,\mathrm{e}^{-\mathrm{i}\kappa(m+\frac{1}{4})}, \\ b_{2,\,m\alpha} &= b_2 + \mathrm{i}\,\delta b_{2\alpha}^+ \mathrm{e}^{\mathrm{i}\kappa(m+\frac{1}{4})} - \mathrm{i}\,\delta b_{2\alpha}^- \,\mathrm{e}^{-\mathrm{i}\kappa(m+\frac{1}{4})}, \\ z_{m\beta} &= m + \delta z_\beta^+ \,\mathrm{e}^{\mathrm{i}\kappa(m-\frac{1}{4})} + \delta z_\beta^- \,\mathrm{e}^{-\mathrm{i}\kappa(m-\frac{1}{4})}, \\ b_{2,\,m\beta} &= b_2 + \mathrm{i}\,\delta b_{2\beta}^+ \mathrm{e}^{\mathrm{i}\kappa(m-\frac{1}{4})} - \mathrm{i}\,\delta b_{2\beta}^- \,\mathrm{e}^{-\mathrm{i}\kappa(m-\frac{1}{4})}. \end{aligned}\right\} \tag{26}$$

It will also be convenient to use perturbation vectors which are defined in terms of 'symmetry' combinations of these variables,

$$\tilde{\boldsymbol{z}} = \{\delta z_\alpha^+ + \delta \bar{z}_\beta^-,\ \delta \bar{z}_\alpha^- + \delta z_\beta^+,\ \delta z_\alpha^+ - \delta \bar{z}_\beta^-,\ \delta \bar{z}_\alpha^- - \delta z_\beta^+\}, \tag{27}$$

$$\tilde{\boldsymbol{b}} = \{\delta b_{2\alpha}^+ + \delta \bar{b}_{2\beta}^-,\ \delta \bar{b}_{2\alpha}^- + \delta b_{2\beta}^+,\ \delta b_{2\alpha}^+ - \delta \bar{b}_{2\beta}^-,\ \delta \bar{b}_{2\alpha}^- - \delta b_{2\beta}^+\}. \tag{28}$$

The evolution equations for these vectors are just a straightforward linear perturbation of the equations of motion (13)–(15), taking into account the contributions from all the vortices in the street.

It is clear, from the structure of (13)–(15), that the perturbation equations will have the general form

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\tilde{\boldsymbol{z}}\\ \tilde{\boldsymbol{b}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_0 + \epsilon^4 \boldsymbol{A}_2 & \boldsymbol{E}_2\,\epsilon^4 \\ \boldsymbol{C}_{-1}/\epsilon^2 + \boldsymbol{C}_1\,\epsilon^2 & \boldsymbol{D}_1/\epsilon^2 + \boldsymbol{D}_1\,\epsilon^2\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{z}}\\ \tilde{\boldsymbol{b}}\end{bmatrix} + \dots \tag{29}$$

The stability of the resulting motion depends on the behaviour of the eigenvalues of (29), which are, in general, slightly perturbed versions of those corresponding to the point-vortex model ($\epsilon = 0$). Because of the presence of the i factor in the left-hand side, the eigenvalues with positive and negative imaginary parts correspond to stable and unstable solutions respectively. Neutrally stable solutions correspond to real eigenvalues. The only relevant change of stability with respect to that model may occur near the critical point at which the point-vortex model is barely stable. It is easy to see that this point is characterized by a real double eigenvalue having a complete set of eigenvectors, and such that the associated Jordan form of the matrix is diagonal (Jimenez 1987). The question is whether, under the perturbation (29), it is still possible to find a double non-degenerate eigenvalue in the neighbourhood of the critical point of original matrix, and it is this region that we will explore here.

If the unperturbed problem is non-degenerate, it is shown in Lancaster (1969) that the eigenvalues of (29) can be expanded as

$$\sigma = \frac{\sigma_{-1}}{\epsilon^2} + \sigma_0 + \sigma_1 \epsilon^2 + \sigma_2 \epsilon^4 + \dots, \tag{30}$$

while the corresponding eigenvectors can be written as columns of a matrix

$$\begin{bmatrix} \boldsymbol{Z} \\ \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_0 + \boldsymbol{Z}_1 \epsilon^2 + \dots \\ \boldsymbol{B}_0 + \boldsymbol{B}_1 \epsilon^2 + \dots \end{bmatrix}, \tag{31}$$

such that
$$\begin{bmatrix} \boldsymbol{A}_0 + \epsilon^4 \boldsymbol{A}_2 & \boldsymbol{E}_2 \epsilon^4 \\ \boldsymbol{C}_{-1}/\epsilon^2 + \boldsymbol{C}_1 \epsilon^2 & \boldsymbol{D}_{-1}/\epsilon^2 + \boldsymbol{D}_1 \epsilon^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{Z} \\ \boldsymbol{B} \end{bmatrix} = \sigma \begin{bmatrix} \boldsymbol{Z} \\ \boldsymbol{B} \end{bmatrix}. \tag{32}$$

If σ is a simple eigenvalue, or a double but defective one, the matrix in (31) will have just one column, while for double non-degenerate eigenvalues, it will be possible to find matrices with two independent columns satisfying (32), and spanning the complete eigenspace of σ.

Expanding (32) in powers of ϵ, we get, to leading order,

$$\sigma_{-1} \boldsymbol{Z}_0 = 0, \quad (\boldsymbol{D}_{-1} - \sigma_{-1} \boldsymbol{I}) \boldsymbol{B}_0 + \boldsymbol{C}_{-1} \boldsymbol{Z}_0 = 0. \tag{33}$$

There are two families of eigenvalues associated to this problem. One of them contains eigenvalues which are $O(1/\epsilon^2)$, have $\sigma_{-1} \neq 0$ and $\boldsymbol{Z}_0 = 0$, and are associated to the standard eigenvalue problem for the matrix $\boldsymbol{D}_{-1}$. They correspond to fast oscillations of the boundary of the vortex cores, and they remain stable to the accuracy of this expansion.

The other family contains eigenvalues of $O(1)$, $\sigma_{-1} = 0$, corresponding to the perturbations of the matrix $\boldsymbol{A}_0$, which is the matrix for the point-vortex model. For these eigenvalues we need the next few members of the perturbation hierarchy,

$$(\boldsymbol{A}_0 - \sigma_0 \boldsymbol{I}) \boldsymbol{Z}_0 = 0, \tag{34a}$$

$$(\boldsymbol{A}_0 - \sigma_0 \boldsymbol{I}) \boldsymbol{Z}_1 = \sigma_1 \boldsymbol{Z}_0, \tag{34b}$$

$$(\boldsymbol{A}_0 - \sigma_0 \boldsymbol{I}) \boldsymbol{Z}_2 = \sigma_1 \boldsymbol{Z}_1 - (\boldsymbol{A}_2 - \sigma_2 \boldsymbol{I}) \boldsymbol{Z}_0 - \boldsymbol{E}_2 \boldsymbol{B}_0. \tag{34c}$$

The top equation in this hierarchy is just the eigenvalue problem for the point-vortex model. As long as σ_0 is not defective it is easy to see (by transforming the problem to the frame of reference formed by the eigenvectors) that $\sigma_1 = 0$, $\boldsymbol{Z}_1 = 0$.

$\boldsymbol{B}_0$ can then be computed from (33), which now becomes

$$\boldsymbol{D}_{-1} \boldsymbol{B}_0 + \boldsymbol{C}_{-1} \boldsymbol{Z}_0 = 0, \tag{35}$$

as $\boldsymbol{B}_0 = -\boldsymbol{D}_{-1}^{-1}\boldsymbol{C}_{-1}\boldsymbol{Z}_0$ and substituted into (34*c*), giving the generalized eigenvalue problem,

$$-(\boldsymbol{A}_0-\sigma_0\boldsymbol{I})\,\boldsymbol{Z}_2 = (\boldsymbol{A}_2-\boldsymbol{E}_2\boldsymbol{D}_{-1}^{-1}\boldsymbol{C}_{-1}-\sigma_2\boldsymbol{I})\,\boldsymbol{Z}_0 \equiv (\boldsymbol{M}-\sigma_2\boldsymbol{I})\,\boldsymbol{Z}_0. \tag{36}$$

This problem can usually be solved only for single particular column vectors $\boldsymbol{Z}_0$, in the original eigenspace of σ_0. The particular case in which the problem can be solved for a two-column matrix, spanning the full original eigenspace, corresponds to the case in which the perturbed matrix also has a double non-degenerate eigenvalue.

5. Stability of the collective mode

In the case of the vortex street, we are interested in that range of the parameters (κ, H), around the critical point (κ_c, H_c), in which the variation of the original matrix compensates the $O(\epsilon^4)$ effects of non-zero area. Since the derivatives of $\boldsymbol{A}$ with respect of those two parameters do not vanish at the critical point, the region of interest can be expected to have a size which is also $O(\epsilon^4)$. Away from this region the eigenvalues of the perturbed problem are just slightly modified versions of those of the point vortex model, and the stability behaviour does not change substantially.

The computation of the perturbation matrices needed in (32) and (36) is difficult to do by hand, but can be readily adapted to automatic symbolic manipulation. As before, we have used REDUCE (Hearn 1983) to obtain the necessary expressions.

From (36) is is clear that we only need to compute the leading term of the evolution equation for $\tilde{\boldsymbol{b}}$, and the first two terms of the equation for $\tilde{\boldsymbol{z}}$. Moreover, if we are only interested in studying the behaviour of the eigenvalues near the critical point, we just need to study the local behaviour of the matrices in that neighbourhood. In particular, all the matrices, except $\boldsymbol{A}_0$, can be evaluated at the critical point, and $\boldsymbol{A}_0$ can be expanded in powers of $\kappa-\kappa_c$ and $H-H_c$, keeping only the linear terms which are absorbed into $\boldsymbol{A}_2$. The result is

$$\boldsymbol{A}_0 = \tfrac{1}{2}\pi^2 \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{M} = \pi^6 \begin{bmatrix} P+a_1 & R & \frac{1}{2}Q & -\frac{1}{12}Q \\ -R & -P+a_2 & \frac{1}{12}Q & -\frac{1}{2}Q \\ \frac{1}{2}Q & -\frac{1}{12}Q & -P-a_2 & R \\ \frac{1}{12}Q & -\frac{1}{2}Q & -R & P-a_1 \end{bmatrix}, \tag{37}$$

where

$$\left.\begin{aligned} P &= \frac{\sqrt{2}-\pi H_c}{2\pi^5\epsilon^4}(\kappa-\pi), \\ R &= \frac{1}{\pi^3\epsilon^4\sqrt{2}}(H-H_c)-\tfrac{1}{8}(1+Q^2), \\ a_1 &= \tfrac{5}{12}(1+Q^2), \\ a_2 &= -\tfrac{1}{12}(1+Q^2). \end{aligned}\right\} \tag{38}$$

The complete expressions for these matrix, at arbitrary values of κ and H, are given in the Appendix. The matrix $\boldsymbol{A}_0$ has two double non-defective eigenvalues. The first observation is that interchanging, in (36), the components $\tilde{z}_1$, $\tilde{z}_2$ with $\tilde{z}_4$, $\tilde{z}_3$, is equivalent to switching the signs of σ_0, σ_2, and $\kappa-\pi$. A consequence is that it is pos-

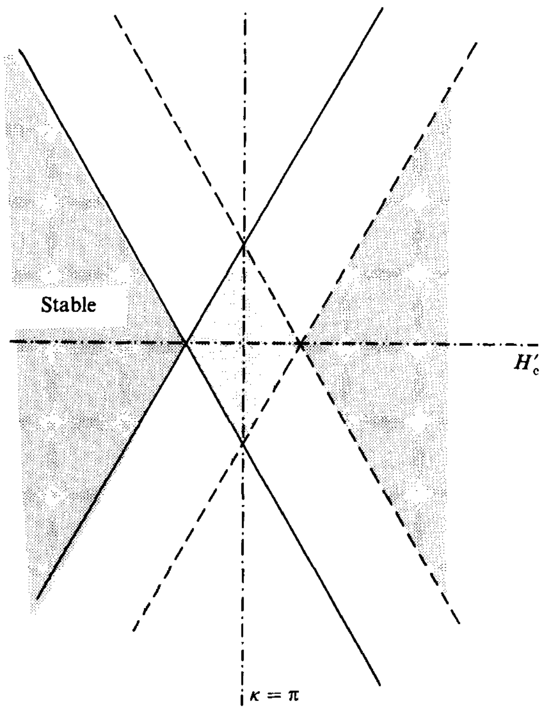

FIGURE 1. Stability diagram for the perturbed vortex street, showing the splitting of the stability 'crossing'. Each of the two symmetric crosses corresponds to a different double eigenvalue.

sible to study only the perturbation of one of the double eigenvalues (e.g. $-\frac{1}{2}\pi^2$), and reverse all the signs for the other one.

For that eigenvalue, the matrix $\boldsymbol{A}_0 - \sigma_0 \boldsymbol{I}$ vanishes identically except for the bottom 2×2 diagonal block, and any vector with its two last components equal to zero is an eigenvector of the unperturbed system. Under those conditions, (36) reduces to a standard eigenvalue problem for the top 2×2 diagonal block in $\boldsymbol{M}$,

$$\begin{bmatrix} P+a_1 & R \\ -R & a_2-P \end{bmatrix} \boldsymbol{v} = \frac{\sigma_2}{\pi^6}\boldsymbol{v}. \tag{39}$$

where $\boldsymbol{v}$ is arbitrary. The condition that (39) has two independent eigenvectors is that $R = 0$ and $a_1 + P = a_2 - P$, which corresponds to values,

$$\kappa'_c - \pi = -\frac{\pi^5(1+Q^2)\,\epsilon^4}{2(\sqrt{2}-\pi H_c)}, \quad H'_c - H_c = \frac{\pi^3(1+Q^2)\,\epsilon^4}{4\sqrt{2}}. \tag{40}$$

At this new critical point the eigenvalue perturbation $\sigma_{2c} = \frac{1}{3}(1+Q^2)$ is still real, and the corresponding solution is neutrally stable. In its neighbourhood, it follows from (39) that the perturbed eigenvalues are

$$\sigma = \sigma'_c \pm \tfrac{1}{2}\pi(\sqrt{2}-\pi H_c)\left[(\kappa-\kappa'_c)^2 - \frac{2\pi^4}{(\sqrt{2}-\pi H_c)^2}(H-H'_c)^2\right]^{\frac{1}{2}}. \tag{41}$$

The situation is the same as for the point-vortex model. The neighbourhood of the critical point is divided into four sectors, alternately stable and unstable. Moreover the line $H = H'_c$, which corresponds to all the modes of a vortex street with just the critical separation between rows, is the only line $H =$ constant which lies completely in the stability region. There are, in fact, two different 'crosses' situated at the same value of H, but symmetrically with respect to $\kappa = \pi$. They separate the (κ, H)-plane into the stability regions shown in figure 1, but do not create any new segment of stability or instability for H. At the point vortex approximation, $(\epsilon = 0)$, both crosses collapse into a single (double) one, but the qualitative stability behaviour does not change.

It is interesting to look in detail into the mechanism by which the cross is preserved in the finite area model. The condition for (39) to have a double non-degenerate eigenvalue is that the matrix be a multiple of the identity. This generates three conditions for the elements of any arbitrary 2×2 matrix, which normally require the degrees of freedom of three parameters to be satisfied. Taking those parameters to be κ, H and either Q or ϵ, we would expect the cross to persist only for a one-dimensional family of combinations of the latter two. Only because of the particular symmetry of the matrix in (39), in which the two off-diagonal terms differ only in sign, is it possible to satisfy the cross with just two independent conditions, and with just two parameters (e.g. κ and H), independently of the other two (Q, ϵ). That is, or an equivalent symmetry, persists to all orders of approximation, when other harmonics are taken into account, is guaranteed by the general arguments in Jiménez (1987).

6. Discussion

We have presented an asymptotic perturbation model for the two-dimensional inviscid vortex street when the point vortices of the Kármán model are substituted by small uniform patches of vorticity. We have shown that, at least to the accuracy of the perturbation expansion, $O(S^2)$, it is possible to have equilibrium vortex streets in which the areas of the vortices in both rows are different. Moreover, even in this case, the stability properties of the street do not change qualitatively with respect to the point model. In particular, all vortex streets contains a finite range of unstable modes, except for a critical separation, which barely avoids instability. The stability diagram is characterized by two diagonal 'crosses' which separate sectors of stable and unstable modes in the (κ, H)-plane. These crosses are located at a common value of H, symmetrically with respect to $\kappa = \pi$, at the points

$$\left.\begin{aligned} \frac{\kappa_c' - \pi}{2\pi} &= \pm \frac{\pi^2}{4(\sqrt{2} - \pi H_c)}(1+Q^2)S^2 \sim 4.63089(1+Q^2)S^2, \\ H_c' - H_c &= \frac{\pi}{4\sqrt{2}}(1+Q^2)S^2 \sim 0.55536(1+Q^2)S^2. \end{aligned}\right\} \tag{42}$$

Therefore, a vortex street whose rows are separated by H_c', is the only one that is barely stable.

This result agrees with the numerical evidence in MSS for streets with rows of equal areas, and with the general arguments in Jiménez (1987). For $Q = 0$, the numerical values in (42) can be compared directly with those obtained in MSS. Their result for the coefficient of the wavenumber perturbation, κ_c', and for the aspect ratio, H_c', are 4.63 and 0.557, in excellent agreement with (42).

It is also interesting to compare our results with those of Kida (1982), who also uses an algebraic expansion of the equations of motion, for the symmetric vortex street, valid to $O(S^2)$. There are two points that need to be considered. First, in Kida, the final results differ from ours in that the stability cross disappears and is substituted by a finite interval of linear stability. This original version, however, contains algebraic errors which were corrected in MSS (p. 190) after a private communication from Kida.

After receiving a preprint of the present paper, Kida, in a private communication, has pointed out that the corrections to his results published in MSS still contain errors, and that the formulae that should be used are, in the notation of Kida (1982),

$$A_1 = A_2 = s(-2EG+2IJ),$$

$$B_1 = -s(E^2+G^2+I^2+J^2-FH), \quad B_2 = s(E^2+G^2+I^2+J^2+FH),$$

$$C_1 = C_2, = -sFK,$$

$$D_1 = s(2EI-FL+2GJ), \quad D_2 = -s(2EI+FL+2GJ),$$

where $s = P^2/2\pi^2 = \frac{1}{2}\epsilon^4$, in our notation.

Once the corrected values are used, the cross reappears, and the stability matrix coincides with ours, as given in the Appendix, for $Q = 0$. There is still a minor discrepancy between our results and those of Kida in that the signs for C_1, C_2 are different in both cases. This sign only appears in the real part of the eigenvalues, and does not influence the position of the stability boundary. It is our belief that it is very difficult to carry out successfully an algebraic expansion of this magnitude without the help of a symbolic manipulation. Even with this help, errors are as difficult to avoid as in numerical programs, and it is important to introduce as many consistency checks as possible. In fact, it was only after comparison with the numerical values in MSS that we were able to locate a (spelling) error in our algebraic program. Most probably, the remaining sign discrepancy derives from one such undetected error either in our expansions or in Kida's.

The second point to be considered is the validity of the expansions in Kida (1982). There are two open questions in this respect. First, Kida uses explicitly a 'coarse grained' approximation in which the second harmonic is substituted by its equilibrium value and its high-frequency internal oscillations are neglected. Secondly, he computes the eigenvalues of the perturbed problem by writing directly the characteristic determinant of the equivalent (32). This implies retaining terms which are $O(S^2)$, even if the elements of the matrix are only valid to $O(S^2)$. In both cases Kida claims that the neglected terms are not important, but offers no proof. Both approximations are justified here. The first one is equivalent to the use of (35), which is just an expression of the equilibrium of the second harmonic. The second one is made precise by the formal expansion in the (30)–(36).

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Appendix

We give here full expressions for the matrices $\boldsymbol{A}_0$ and $\boldsymbol{M}$ at an arbitrary point, not necessarily close to the critical one. We also give expressions for the various functions which appear in the matrix elements and which are defined in terms of infinite sums similar to the one in (20). Whenever the same symbol is defined here and in the body of the paper, both definitions are identical, and have just been recast in terms of different basic variables. Also, when symbols are defined using the $\pm$ notation, the upper sign corresponds to the first of the two symbols separated by commas in the left-hand side of the equation. The matrix $\boldsymbol{A}_0$ takes the form

$$\boldsymbol{A}_0 = \begin{bmatrix} s_{2m} & -\mu_0 & 0 & 0 \\ \mu_0 & s_{2p} & 0 & 0 \\ 0 & 0 & -s_{2m} & -\mu_0 \\ 0 & 0 & \mu_0 & -s_{2p} \end{bmatrix}.$$

The elements for $\boldsymbol{M}$ are,

$$M_{11} = 2(1+Q^2)\,[(\tau_3+s_3)^2+s_{3m}^2-3s_{4m}\,b_2],$$
$$M_{22} = -2(1+Q^2)\,[(\tau_3-s_3)^2+s_{3p}^2+3s_{4p}\,b_2],$$
$$M_{33} = 2(1+Q^2)\,[(\tau_3+s_3)^2+s_{3m}^2+3s_{4m}\,b_2],$$
$$M_{44} = -2(1+Q^2)\,[(\tau_3-s_3)^2+s_{3p}^2-3s_{4p}\,b_2],$$
$$M_{13} = M_{31} = 4Q[(\tau_3+s_3)^2-s_{3m}^2],$$
$$M_{24} = M_{42} = -4Q[(\tau_3-s_3)^2-s_{3p}^2],$$
$$M_{12}, M_{34} = -M_{21}, -M_{43} = \pm 2(1+Q^2)\,[s_{3m}(\tau_3-s_3)-s_{3p}(\tau_3+s_3)\pm 3A_4\,b_2],$$
$$M_{14}, M_{23} = -M_{41}, -M_{32} = 4Q[s_{3m}(\tau_3-s_3)+s_{3p}(\tau_3+s_3)\pm 3(A_4+\tau_4)\,b_2],$$

where the functions used in the definitions are

$$\tau_3 = -\pi^3\theta(\theta^2-1), \quad \tau_4 = \tfrac{1}{3}\pi^4(\theta^2-1)(3\theta^2-1), \quad s_3 = \tfrac{1}{6}\pi^3\phi(\phi^2-1),$$
$$\mu_0 = \tfrac{1}{2}\pi^2(\phi^2+1-2\theta^2), \quad A_4 = \tfrac{1}{24}\pi^4(\phi^2-1)^2-\tau_4, \quad b_2 = \tfrac{1}{3}\pi^2(3\theta^2-2),$$
$$s_{2p}, s_{2m} = \pi^2\,\mathrm{e}^{\mp\pi H\phi}(1-\theta^2)^{\frac{1}{2}}(\theta\pm\phi),$$
$$s_{3p}, s_{3m} = -\pi^3\,\mathrm{e}^{\mp\pi H\phi}(1-\theta^2)^{\frac{1}{2}}(\theta^2\pm\theta\phi+\tfrac{1}{2}(\phi^2-1)),$$
$$s_{4p}, s_{4m} = \pi^4\,\mathrm{e}^{\mp\pi H\phi}(1-\theta^2)^{\frac{1}{2}}(\theta^3\pm\theta^2\phi+\tfrac{1}{6}(3\phi^2-5)\theta\pm\tfrac{1}{6}\phi(\phi^2-3),$$
$$\theta = \tanh(\pi H), \phi = (\kappa-\pi)/\pi.$$

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